Dawson college
Department of Mathematics
201 NYB 05
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9:30 AM to 12:30 PM
5/25/22
Final exam-A

Student Name: ____________________
Student ID: ____________________

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Instructions

1. Write your name and ID number at the top, right of this page.
2. Please note that there are 14 questions.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use the back of the page and refer to this page in your solutions.
4. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions. Numerical answers should be in exact values.**
5. **DO NOT WRITE FORMULAS ON THIS COVER PAGE.**

Marking Scheme:

Question	**Out of**	**Score**
1	6	
2	16	
3	6	
4	8	
5	6	
6	6	
7	6	
8	10	
9	3	
10	3	
11	3	
12	16	
13	5	
14	6	
$\sum$	100	

1. Use a limit of Riemann sums to compute the exact value of the definite integral (6 marks)

$$\int_0^2 (4 - x + x^3)\,dx$$

$$\Delta x = \frac{2}{n}, \quad x_i = \frac{2i}{n}$$

$$I = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - \frac{2i}{n} + \frac{8i^3}{n^3}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \left(8 - \frac{4}{n^2}\frac{n(n+1)}{2} + \frac{16}{n^4}\frac{n^2(n+1)^2}{4}\right)$$

$$= 8 - 2 + 4 = 10$$

$$\sum_{i=1}^{n} c = cn, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

2. Evaluate the following integrals (16 marks)

(a)

$$I = \int x(\ln(4x))^2\,dx$$

$$u = (\ln(4x))^2 \quad , \quad x\,dx = dv$$

$$du = 2\ln(4x)\cdot\frac{1}{x}\,dx \quad , \quad v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2}(\ln(4x))^2 - \int \frac{x^2}{2}\,2\ln(4x)\,\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}(\ln(4x))^2 - \int x\ln(4x)\,dx$$

$$u = \ln(4x) \quad , \quad x\,dx = dv$$

$$du = \frac{1}{x}\,dx \quad ; \quad v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2}(\ln(4x))^2 - \frac{x^2}{2}\ln(4x) + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}(\ln(4x))^2 - \frac{x^2}{2}\ln(4x) + \frac{1}{4}x^2 + C$$

(b)

$$\int \sin^4(2x)\,dx$$

$$I = \int \left(\sin^2(2x)\right)^2 dx$$

$$= \int \left(\frac{1}{2}(1 - \cos(4x))\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(4x) + \cos^2(4x)\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(4x) + \frac{1}{2}(1 + \cos(8x))\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(4x) + \frac{1}{2}\cos(8x)\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \frac{1}{2}\sin(4x) + \frac{1}{16}\sin(8x)\right) + C$$

(c)

$$\int_1^4 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$$

$1+\sqrt{x}=t \Rightarrow \sqrt{x}=t-1$

$\frac{dx}{2\sqrt{x}} = dt$ or $dx = 2(t-1)\,dt$

$$\int_1^4 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \int_2^3 \frac{t-1}{t}\,2(t-1)\,dt$$

$$= 2\int_2^3 \left(t-2+\frac{1}{t}\right)dt$$

$$= 2\left[\frac{t^2}{2} - 2t + \ln|t|\right]_2^3$$

$$= 2\left(\frac{9}{2} - 6 + \ln 3 - 2 + 4 - \ln 2\right)$$

$$= \left(\frac{1}{2} + \ln\frac{3}{2}\right)2$$

$$= 1 + 2\ln\frac{3}{2}$$

(d)

$$\int \frac{2x^3 + x^2 + 2x - 3}{x^4 + x^2} \, dx$$

$$= \int \left(\frac{2}{x} - \frac{3}{x^2} + \frac{4}{x^2+1} \right) dx$$

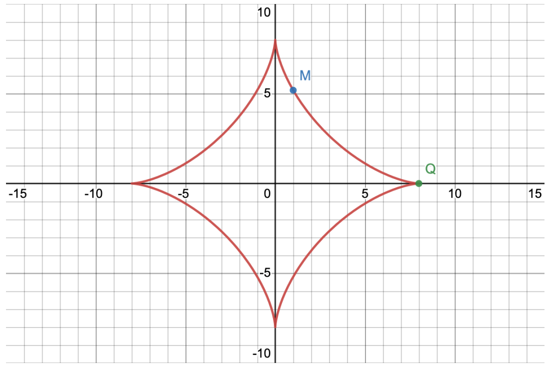
$$= 2 \ln|x| + \frac{3}{x} + 4 \arctan(x) + C$$

3. (6 marks)

The equation

$$x^{2/3} + y^{2/3} = 4$$

represents a special polygon in mathematics called Astroid. Find the are length, located in the first quadrant, from $M(1, \sqrt{27})$ to $Q(8, 0)$.



$$y^{2/3} = 4 - x^{2/3}$$

$$y = \left(4 - x^{2/3}\right)^{3/2}$$

$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}\right)x^{-\frac{1}{3}}$$

$$= -x^{-1/3}\left(4 - x^{2/3}\right)^{1/2}$$

$$(y')^2 = x^{-2/3}\left(4 - x^{2/3}\right) = 4x^{-\frac{2}{3}} - 1$$

$$L = \int_{x=1}^{8} \sqrt{1 + 4x^{-2/3} - 1}\, dx = \int_{1}^{8} 2x^{-\frac{1}{3}}\, dx$$

$$= 2\left.\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right|_{1}^{8}$$

$$= 3\left[(8)^{2/3} - 1^{2/3}\right] = 9$$

4. Let f be a continuous function and $\int_1^{\sqrt{2}} f(x)\,dx = 8$. (8 marks)

(a) Evaluate $\int_0^{\pi/4} \sin(\theta) f(\sqrt{2}\cos(\theta))\,d\theta$.

$$\int_0^{\pi/4} \sin(\theta) f(\sqrt{2}\cos\theta)\,d\theta$$

$\sqrt{2}\cos\theta = x$

$dx = -\sqrt{2}\sin\theta\,d\theta$

$\theta = 0 \Rightarrow x = \sqrt{2}$

$\theta = \frac{\pi}{4} \Rightarrow x = 1$

$$= -\frac{1}{\sqrt{2}} \int_{\sqrt{2}}^{1} f(x)\,dx = \frac{8}{\sqrt{2}} = \boxed{4\sqrt{2}}$$

(b) Suppose that $f(1) = 2$, $f(\sqrt{2}) = 6$. Evaluate $\int_1^{\sqrt{2}} x f'(x)\,dx$.

$$\int_1^{\sqrt{2}} x f'(x)\,dx$$

$f'(x)\,dx = dv$

$v = f(x)$

$u = x \Rightarrow du = dx$

$$= \left[\left[x f(x) \right]_1^{\sqrt{2}} - \int_1^{\sqrt{2}} f(x)\,dx \right]$$

$$= \left[\sqrt{2} f(\sqrt{2}) - f(1) - 8 \right]$$

$$= 6\sqrt{2} - 2 - 8 = 6\sqrt{2} - 10$$

5. Determine by calculation whether the following improper integral is convergent or divergent. (6 marks)

$$\int_1^2 \frac{1}{\sqrt{2x - x^2}}\,dx$$

$$\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$= \arcsin(x-1)$$

$$I = \lim_{t \to 2^-} \int_1^t \frac{1}{\sqrt{2x-x^2}}\,dx$$

$$= \lim_{t \to 2^-} \Big[\arcsin(x-1)\Big]_1^t$$

$$= \lim_{t \to 2^-} \left[\arcsin(t-1) - \arcsin(0)\right]$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

6. Determine by calculation whether the following improper integral is convergent or divergent. (6 marks)

$$\int_1^\infty \frac{1}{x(x+1)}\,dx$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + Bx$$

$$A = 1 \ , \ B = -1$$

$$\int_1^\infty \frac{1}{x(1+x)}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x(x+1)}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \lim_{t\to\infty} \left[\ln t - \ln t+1 + \ln 2\right]$$

$$= \lim_{t\to\infty} \left[\ln \frac{t}{t+1} + \ln 2\right]$$

$$= \ln 2$$

7. The speed of a particle, moving on a straight line, is given by $v(t) = e - \ln t^e$. Evaluate the integral $\int_1^3 |v(t)|\, dt$ (which represents the traveled distance of the particle between $t = 1$ to $t = 3$). (6 marks)

$$v(t) = e - \ln t^e = 0$$

$$= e - e\ln t = 0$$

$$\ln t = 1 \Rightarrow t = e$$

$$\int (e - e\ln t)\, dt = et - e(t\ln(t) - t)$$

$$= 2et - et\ln(t)$$

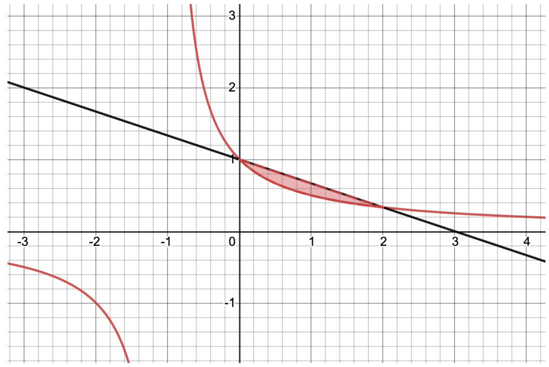
$$\int_1^3 |v(t)|\, dt = \int_1^e (e - e\ln t)\, dt + \int_e^3 -(e - e\ln t)\, dt$$

$$= \Big[2et - et\ln(t)\Big]_1^e - \Big[2et - et\ln(t)\Big]_e^3$$

$$= 2e^2 - e^2 - 2e - \big[6e - 3e\ln(3) - 2e^2 + e^2\big]$$

$$= 2e^2 - 8e + 3e\ln(3)$$

8. Let R be the shaded region enclosed by $y = \frac{1}{x+1}$, $y = 1 - \frac{x}{3}$. See below (10 marks)



(a) Set up an integral or integrals that express the area of R (DO NOT EVALUATE).

$$A = \int_0^2 \left(1 - \frac{x}{3} - \frac{1}{x+1}\right) dx$$

(b) Use the method of your choice(Disk-Washer method or Shell method) to set up the integral to find the volume obtained by rotating region R around $x = 3$ (DO NOT EVALUATE)

$$V = 2\pi \int_0^2 (3-x)\left(1 - \frac{x}{3} - \frac{1}{x+1}\right) dx$$

(c) Use the method of your choice(Disk-Washer method or Shell method) to set up the integral to find the volume obtained by rotating region R around $y = 2$ axis (DO NOT EVALUATE)

$$V = \pi \int_0^2 \left(\left(2 - \frac{1}{x+1}\right)^2 - \left(2 - 1 + \frac{x}{3}\right)^2\right) dx$$

9. Determine whether the sequence $\{\frac{3^n}{(n+1)^2}\}$ is convergent or divergent. (3 marks)

$$\lim_{x\to\infty} \frac{3^x}{(x+1)^2} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{H}{=}} \lim_{x\to\infty} \frac{3^x \ln 3}{2(x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{3^x (\ln 3)^2}{2} = \infty$$

and $\left\{\frac{3^n}{(n+1)^2}\right\}$ div.

10. Express the repeating decimal 7.162162... as a rational number by using a geometric series. (3 marks)

$$7 + 0.\overline{162}$$

$$0.162162162\cdots = \frac{162}{1000} + \frac{162}{(1000)^2} + \frac{162}{(1000)^3} + \cdots$$

geo. with $a = \frac{162}{1000}$, $r = \frac{1}{1000}$

$$S = \frac{a}{1-r} = \frac{\frac{162}{1000}}{1-\frac{1}{1000}} = \frac{162}{999}$$

and $7.\overline{162} = 7 + \frac{162}{999} = \frac{7155}{999}$

11. Determine whether the series is convergent or divergent, if it is convergent find its sum.

(3 marks)

$$\sum_{n=1}^{+\infty}\Big(\tan(\frac{1}{n})-\tan(\frac{1}{n+1})\Big)$$

$$S_1=\tan(1)-\tan(\tfrac{1}{2})$$

$$S_2=\tan(1)-\cancel{\tan(\tfrac{1}{2})}+\cancel{\tan(\tfrac{1}{2})}-\tan(\tfrac{1}{3})$$

$$=\tan(1)-\tan(\tfrac{1}{3})$$

$$S_n=\tan(1)-\tan(\tfrac{1}{n+1})$$

$$\lim S_n=\lim_{n\to\infty}\Big(\tan(1)-\tan(\tfrac{1}{n+1})\Big)$$

$$=\tan(1)$$

$$\sum_{n=1}^{\infty}\Big(\tan(\tfrac{1}{n})-\tan(\tfrac{1}{n+1})\Big)\text{ conv and } S=\tan(1)$$

12. Determine whether the following series is absolutely convergent, conditionally convergent or divergent, state clearly the test you have used in each problem. (16 marks)

(a)

$$\sum_{n=2}^{+\infty} \frac{(-1)^n}{e^{\sqrt{n}}}$$

the series of abs values $\displaystyle\sum_{n=2} \frac{1}{e^{\sqrt{n}}}$

$f(x) = \dfrac{1}{e^{\sqrt{x}}} = e^{-\sqrt{x}}$ is Continuous, dec, + on $[2, +\infty)$

$$\int e^{-\sqrt{x}}\,dx = 2\int t\,e^{-t}\,dt \qquad u = t,\ e^{-t}dt = dv \quad du = dt,\ -e^{-t} = v$$

$$= 2\left[-te^{-t} + \int e^{-t}\,dt\right]$$

$$= 2\left[-te^{-t} - e^{-t}\right]$$

$$= 2\left[-\sqrt{x}\,e^{-\sqrt{x}} - e^{-\sqrt{x}}\right]$$

$$\int_2^{\infty} e^{-\sqrt{x}}\,dx = \lim_{t\to\infty} 2\left[-\sqrt{x}\,e^{-\sqrt{x}} - e^{-\sqrt{x}}\right]_2^t$$

$$= \lim_{t\to\infty} 2\Big(\underbrace{-\sqrt{t}\,e^{-\sqrt{t}}}_{\to 0}{}^{*} - \underbrace{e^{-\sqrt{t}}}_{\to 0}\Big) - 2\left(-\sqrt{2}\,e^{-\sqrt{2}} - e^{-\sqrt{2}}\right)$$

$$= 2\left(\sqrt{2}\,e^{-\sqrt{2}} + e^{-\sqrt{2}}\right)$$

and $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-\sqrt{n}}$ is abs Conv

$$* \quad \lim_{t\to\infty} \frac{\sqrt{t}}{e^{\sqrt{t}}} \overset{H}{=} \lim_{t\to\infty} \frac{\frac{1}{2\sqrt{t}}}{\frac{1}{2\sqrt{t}}\,e^{\sqrt{t}}} = 0$$

(b)

$$\sum_{n=1}^{+\infty}\left(\frac{\sqrt{4n^2+1}}{n}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{\sqrt{4n^2+1}}{n}\right)^n}=\lim_{n\to\infty}\frac{\sqrt{4n^2+1}}{n}$$

$$=\lim_{n\to\infty}\frac{\sqrt{4+\frac{1}{n^2}}}{1}=2>1$$

by root test $\sum_{n=1}\left(\frac{(4n^2+1)^{1/2}}{n}\right)^n$ is div.

(c)

$$\sum_{n=1}^{+\infty}\frac{n!}{n^{n+1}}$$

$$\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}}=\lim_{n\to\infty}\frac{(n+1)\,\cancel{n!}\;n^n\;n}{(n+1)^n\,(n+1)^2\;\cancel{n!}}$$

$$=\lim_{n\to\infty}\underbrace{\frac{n(n+1)}{(n+1)^2}}_{\to 1}\;\underbrace{\frac{n^n}{(n+1)^n}}_{\to\frac{1}{e}}$$

$$=\frac{1}{e}<1$$

student must provide details here

by ratio test

$\sum_{n=1}\frac{n!}{n^{n+1}}$ is Conv

(d)

$$\sum_{n=1}^{+\infty} \frac{(-1)^n \, 3n^2}{n^3+1}$$

the series of abs values $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$

$$\lim_{n\to\infty} \frac{\frac{3n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3 \neq 0$$

Since $\sum \frac{1}{n}$ div. (harmonic Series)

$\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$ is div (Lim. Comp. test)

(no abs. Conv.)

We have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \, 3n^2}{n^3+1}$

$$b_n = \frac{3n^2}{n^3+1} \longrightarrow 0 \text{ as } n \to \infty$$

$$f(x) = \frac{3x^2}{x^3+1}, \quad f'(x) = \frac{-3x(x^3-2)}{(x^3+1)^2} < 0 \text{ for } x > \sqrt[3]{2}$$

thus b_n is decreasing for $n \geq 2$.

by Alt. Series test, the series $\displaystyle\sum \frac{(-1)^n \, 3n^2}{n^3+1}$ is Conv.

therefore $\displaystyle\sum \frac{(-1)^n \, 3n^2}{n^3+1}$ is Conditionally convergent.

13. Find the radius of convergence and the interval of convergence of (5 marks)

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)(n+2)}(x+1)^n$$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+2)(n+3)}|x+1|^{n+1}}{\frac{1}{n(n+1)(n+2)}|x+1|^n}$$

$$= \lim_{n\to\infty} \frac{n(n+1)(n+2)}{(n+1)(n+2)(n+3)}|x+1| = |x+1| < 1$$

the radius of Conv. is $R=1$

the series Centered at $C=-1$

At $x=0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ direct Comp. with $\sum \frac{1}{n^3}$

Conv.

At $x=-2$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)(n+2)}$ Alt. Series test

interval of Conv. $[-2, 0]$

14. Find the Taylor series of $f(x) = \frac{1}{\sqrt{x}}$ about $a = 4$. (6 marks)

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{-\frac{1}{2}}$	$\frac{1}{2}$
1	$-\frac{1}{2} x^{-\frac{3}{2}}$	$-\frac{1}{2} \cdot \frac{1}{2^3}$
2	$\frac{1}{2} \frac{3}{2} x^{-\frac{5}{2}}$	$\frac{1}{2} \frac{3}{2} \frac{1}{2^5}$
3	$-\frac{1}{2} \frac{3}{2} \frac{5}{2} x^{-\frac{7}{2}}$	$-\frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{1}{2^7}$
4	$+\frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{7}{2} x^{-\frac{7}{2}}$	$\frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{7}{2} \frac{1}{2^9}$
n		$(-1)^n \frac{(1)(3) \cdots (2n-1)}{2^{3n+1}}$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{f^{(n)}(4)}{n!} (x-4)^n$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3) \cdots (2n-1)}{2^{3n+1}} (x-4)^n$$