# DAWSON COLLEGE
# MATHEMATICS DEPARTMENT
## Linear Algebra (SCIENCE)

**I confirm that I have read and understood the College's Academic Integrity Document and will adhere to the principles of academic integrity while writing this exam.**

**201-NYC-05 S01-07**
**Fall 2022**
**Final Examination**
**December** $13^{th}$**, 2022**
**Time Limit: 3 hours**

Name: ______________________

ID#: ______________________

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- This exam contains 15 pages (including this cover page) and 16 problems. Check to see if any pages are missing.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your solutions in a legible and orderly manner.
- You are only permitted to use the **Sharp EL-531**** calculator.
- Good luck!

| Question | Points | Score |
|---|---|---|
| 1 | 9 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 8 | |
| 5 | 8 | |
| 6 | 3 | |
| 7 | 4 | |
| 8 | 19 | |
| 9 | 4 | |
| 10 | 5 | |
| 11 | 5 | |
| 12 | 4 | |
| 13 | 2 | |
| 14 | 8 | |
| 15 | 5 | |
| 16 | 6 | |
| Total: | 100 | |

1. (a) (5 marks) Solve by Gauss Jordan Elimination

$$\begin{aligned} x_2 - 2x_3 - 2x_4 - x_5 &= 7 \\ -3x_1 + 2x_2 - x_3 - 6x_4 \quad &= -5 \\ x_1 - x_2 \quad + 2x_4 - x_5 &= 2 \end{aligned}$$

(b) (1 mark) Give a particular solution.

(c) (3 marks) Find the solutions of the homogeneous system having the same coefficients matrix.

2. (5 marks) Find the values of $a$ for which the following system

$$\begin{cases} x + y + 3z = 2 \\ -3x + (a-1)y - 10z = -7 \\ 4x + (2-a)y + (a^2-8)z = a-2 \end{cases}$$

has
(1) exactly one solutions,
(2) infinitely many solutions,
(3) no solutions.

3. (5 marks) Solve for $X$ the following equation:

$$(AX+3I)^{-1}C = BA^{-1}C^{T}(BA^{T}C^{-1})^{T}$$

where $A = \begin{bmatrix} 13 & 5 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$.

4. (a) (4 marks) Find the inverse of the matrix $A$ using the inversion algorithm:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ -2 & -2 & -11 \end{bmatrix}$$

(b) (2 marks) Solve for $x$, $y$, $z$, where $\begin{bmatrix} x & y & z \end{bmatrix} A = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ using the $A^{-1}$ found in (a).

(c) (2 marks) Find two elementary matrices $E_1$ and $E_2$ which satisfy $E_2E_1A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

5. Given that:

$$\det A = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 2 \quad ; \quad B = \begin{bmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

(a) (4 marks) Find $\det B$.

(b) (4 marks) Find $\det\left(5A^4(A^{-1})^T\operatorname{adj}(A)\right)$.

6. (3 marks) Solve the linear system using Cramer's rule:

$$\begin{cases} 12x+5y=1 \\ 5x+2y=-1 \end{cases}.$$

7. (4 marks) Assume that a square matrix $A$ satisfies $2A^2+5A-4I=0$. Show that $2A-I$ is invertible and find its inverse in terms of $A$.

8. Consider the vectors $\mathbf{u} = (2,0,1)$, $\mathbf{v} = (-1,-1,1)$ and the points $P(7,1,6)$ and $Q(5,-1,-6)$ in $\mathbb{R}^3$.

   (a) (4 marks) Find the general equation of the plane parallel to **u** and **v** and passing through $P$.

   (b) (4 marks) Determine whether the lines $L_1 : \begin{cases} x = 7+2s \\ y = 1 \\ z = 6+s \end{cases}$ and $L_2 : \begin{cases} x = 5-t \\ y = -1-t \\ z = -6+t \end{cases}$ are parallel, skew, interscting or coinciding.

(c) (4 marks) Find the distance between the point $P(7,1,6)$ and the line $L_2: \begin{cases} x = 5 - t \\ y = -1 - t \\ z = -6 + t \end{cases}$.

(d) (5 marks) Find the two points on the lines $L_1: \begin{cases} x = 7 + 2s \\ y = 1 \\ z = 6 + s \end{cases}$ and $L_2: \begin{cases} x = 5 - t \\ y = -1 - t \\ z = -6 + t \end{cases}$ that are closest to each other.

(e) (2 marks) Given the lines $L_1 : \begin{cases} x = 7+2s \\ y = 1 \\ z = 6+s \end{cases}$ and $L_2 : \begin{cases} x = 5-t \\ y = -1-t \\ z = -6+t \end{cases}$, find the parametric equations of the line that intersects both $L_1$ and $L_2$ at right angles.

9. (4 marks) Let $\mathbf{u}$, $\mathbf{v}$ be unit vectors in $\mathbb{R}^n$ and assume that they are all orthogonal to each other. Simplify: $\text{Proj}_{\mathbf{u}+\mathbf{v}}(\mathbf{u}-2\mathbf{v})$.

10. (5 marks) Consider the vectors in $\mathbb{R}^3$: $\mathbf{u}(\theta) = (\cos\theta, \sin\theta, 0)$ and $\mathbf{v} = (1,0,1)$. Find all the values of the angle $\theta$ in $[0, 2\pi)$ for which the parallelepiped spanned by $\mathbf{u}(\theta)$, $\mathbf{v}$ and $\mathbf{u}(\theta) \times \mathbf{v}$ has volume $V = 2$.

11. Let $\mathbf{u}$ be a unit vector, and let $\mathbf{v}$ be a vector such that $||\mathbf{v}|| = 3$, and $||2\mathbf{u} - \mathbf{v}|| = \sqrt{19}$.

    (a) (4 marks) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

    (b) (1 mark) Find $||\mathbf{u} \times \mathbf{v}||$.

12. (4 marks) Consider the vector space $\mathbf{P}_2$ of polynomials of degree $\leq 2$ and the three polynomials

$$p_1 = 1 + \lambda x \quad p_2 = x + 2x^2 \quad p_3 = 3 - x^2$$

Find the values of $\lambda$ that make $S = \{p_1, p_2, p_3\}$ linearly dependent.

13. (2 marks) Consider the vectors $\mathbf{u} = (1, 2, -3)$ and $\mathbf{v} = (0, 7, 1)$ where $\mathscr{B} = \{\mathbf{u}, \mathbf{v}\}$ is a basis for a vector space $W$. Find the coordinates of the vector $\mathbf{w} = (1, 23, 0)$ relative to $\mathscr{B}$.

14. Consider the space $M_{3\times3}$ of all square matrices of size $3\times3$ and the subset $W$ of skew-symmetric matrices:

$$W = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \ a, b, c \text{ in } \mathbb{R} \right\}$$

(a) (2 marks) Show that $W$ is a subspace of $M_{3\times3}$.

(b) (5 marks) Consider the matrices:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Determine whether $S = \{A, B, C\}$ is a basis for $W$.

(c) (1 mark) Determine the dimension of $W$.

15. (5 marks) Consider the matrices $A$, $B$, $C$, all square and of the same size. Assume that the linear systems $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$ have only the trivial solution and that $C$ is row equivalent to $B$. Prove that $AC$ can be written as a product of elementary matrices.

16. Determine whether each of the following statements is true or false. If the statement is false, provide a counterexample. If the statement is true, provide a proof of the statement.

    (a) (3 marks) If $\det(A + A^T) = 0$, then $A$ is singular.

    (b) (3 marks) If $\mathbf{u}, \mathbf{v}, \mathbf{p}, \mathbf{q}$ be nonzero vectors in $\mathbb{R}^3$ and $\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{p}, \mathbf{q}\}$ then $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

**Answers:**

1. a: $\begin{cases} x_1 = \frac{11}{3} - \frac{4}{3}s - \frac{2}{3}t \\ x_2 = \frac{5}{3} + \frac{2}{3}s - \frac{5}{3}t \\ x_3 = \frac{8}{3} - \frac{2}{3}s - \frac{4}{3}t \\ x_4 = s \\ x_5 = t \end{cases}$ ; b: $t = 0, s = 0 \to (x_1, x_2, x_3, x_4, x_5) = \left(\frac{11}{3}, \frac{5}{3}, \frac{8}{3}, 0, 0\right)$; c: $\begin{cases} x_1 = -\frac{4}{3}s - \frac{2}{3}t \\ x_2 = \frac{2}{3}s - \frac{5}{3}t \\ x_3 = -\frac{2}{3}s - \frac{4}{3}t \\ x_4 = s \\ x_5 = t \end{cases}$

2. one solution: $a \neq \left\{-2, \pm\sqrt{21}\right\}$; infinitely many solutions: does not occur; no solutions: $a = \left\{-2, \pm\sqrt{21}\right\}$

3. $X = \begin{bmatrix} -16 & 53/4 \\ 42 & -137/4 \end{bmatrix}$

4. a: $A^{-1} = \begin{bmatrix} -1 & -2 & -1 \\ -10 & -9 & -5 \\ 2 & 2 & 1 \end{bmatrix}$; b: $x = 3$, $y = 4$, $z = 2$; c: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

5. a: $\det(B) = 30$; b: 4000

6. $(-7, 17)$

7. $(2A - I)^{-1} = A + 3I$

8. a: $x - 3y - 2z + 8 = 0$; b: skew; c: $14\sqrt{6}/3$; d: $R_1(-1, 1, 2)$ in $L_1$, $R_2(1, -5, -2)$ in $L_2$; e: $x = 1 + 2r$, $y = -5 - 6r$, $z = -2 - 4r$, $r$ in $\mathbb{R}$

9. $-\frac{1}{2}(\mathbf{u} + \mathbf{v})$

10. $\theta = \pi/2$, $3\pi/2$

11. a: $\theta = 2\pi/3$; b: $3\sqrt{3}/2$

12. $\lambda = 1/6$

13. $(\mathbf{w})_{\mathscr{B}} = (1, 3)$

14. a: Axioms 1 and 6 are satisfied; b: yes, it is; c: $\dim(W) = 3$

15. Use the Equivalence Theorem and the properties of matrices.

16. a: False; b: True.