DAWSON COLLEGE
Mathematics Department
Final Examination
Linear Algebra
201–NYC–05 (Commerce)
Fall 2021

1. a) (5 marks) Use Gauss-Jordan elimination to find the general solution of the system.
   b) (1 mark) Find a particular solution in which $x_2 = 5$ and $x_3 = 0$.

$$\begin{aligned} -x_1 - 2x_2 + 2x_3 - 4x_4 &= -3 \\ 3x_1 + 6x_2 - 5x_3 + 14x_4 &= 1 \\ 4x_1 + 8x_2 - 6x_3 + 20x_4 &= -4 \end{aligned}$$

2. (6+4 marks) Given the system of linear equations $\begin{cases} 2x - y - 3z = 3 \\ -x + 4y + z = 6 \\ 2x + 4y - z = 11 \end{cases}$.

   a) Solve the system using the inverse matrix. Use the **adjoint** matrix to find the inverse.

   b) Use Cramer's rule to solve the system **<u>for "y" only</u>**.

3. (4 marks) Let $A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 4 & 2 \\ 1 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Solve for X : $\left(CX^{-1} + I\right)^{-1} = XA^T B$.

4. (3 marks) If $A = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 7 & 7 \end{bmatrix}$, find elementary matrices E_1 and E_2 such that $E_2 E_1 A = B$.

5. (3 marks) Simplify as much as possible $B\left(\left(2C^{-1}B\right)^{-1} \cdot \left(D^T C\right)^{-1} \cdot \left(4A^T D\right)^T + I\right) \cdot \left(BD^0 + 2A\right)^{-1}$.

6. (4 marks) Determine the values of a such that the system has
   1) a unique solution, 2) infinitely many solutions, 3) no solution :

$$\begin{cases} x + 2y + 4z = 3 \\ \quad y - 7az = 2 \\ -x - 3y + \left(a^2 + 2\right)z = a - 6 \end{cases}$$

7. (3 marks) Let A be an invertible skew-symmetric $n \times n$ matrix and B be a symmetric $n \times n$ matrix.
   Is a matrix $X = A^{-1}BA^2 + A^2BA^{-1}$ symmetric, skew-symmetric or neither?

8. (4 marks) Evaluate the determinant by a combination of row operations and cofactor expansion
   (at least one row operation must be performed).

$$\begin{vmatrix} 1 & 2 & -4 & 2 \\ 2 & 3 & 8 & 7 \\ 1 & -1 & -3 & 2 \\ 3 & -5 & -2 & 6 \end{vmatrix}$$

9. (4+3+3 marks) If $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A)=-2$, then find

a) $\begin{vmatrix} 2d & a+g & -g+a \\ 6e & 3b+3h & -3h+3b \\ 2f & c+i & -i+c \end{vmatrix}$

b) $\det\left((3A)^2\cdot\det(A^{-1})\right)$

c) $\det\left(4A^{-1}+adj(2A)\right)$

10. (3+3+3 marks) Let $\vec{u}=(4,\ 2,-3)$, $\vec{v}=(2,-1,\ x)$ and $\vec{w}=(-3,\ 1,-2)$.

a) Find the value(s) of x such that the vector $\vec{v}-\vec{u}$ is perpendicular to $\vec{v}+\vec{u}$.

b) Find $Proj_{\vec{w}}\left(2\vec{u}\right)$

c) For which value(s) of x do the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ lie in the same plane when positioned so that their initial points coincide?

11. (3+3 marks) If $\left(4\vec{a}+5\vec{b}\right)\times\left(2\vec{a}+3\vec{b}\right)=(2,4,6)$, then

a) Find the area of the triangle determined by $\vec{a}$ and $\vec{b}$.

b) Find all unit vectors perpendicular to $\vec{a}$ and $\vec{b}$.

12. (3+3 marks) Consider two planes $-x+y-2z=1$ and $3x-2y+z=2$

a) Find the parametric equations of the line of intersection of these planes.

b) Find parametric equations of the line that passes through the point $A(2,-1,3)$ and is parallel to both planes.

13. (3+3 marks) Given a point $A(2,-1,\ 3)$ and a line $\begin{cases} x=-1+t \\ y=1-2t \\ z=2+2t \end{cases}$.

a) Find the distance from the point A to the line (without finding the closest point).

b) Find a point on the line which is closest to the point A.

14. (3 Marks) Determine whether the following statement is true or false. <u>If the statement is true, then prove it.</u> <u>If the statement is false, then provide a counterexample that shows that the statement is not true.</u>

"Let A and B be (2021×2021) matrices, then if $A^TB+BA^T=0$ then at least one matrix A or B is not invertible".

15. (3+3+3 marks) Suppose L1: $\begin{cases} x=-1+t \\ y=1-2t \\ z=3+2t \end{cases}$ and L2: $\begin{cases} x=3+2u \\ y=2-u \\ z=1+u \end{cases}$.

a) Show that L1 and L2 are skew lines.

b) Find the distance between these lines.

c) Find the equation of the plane which contains the line L1 and is parallel to the line L2.

16. (7 marks) Maximize $P = 2x_1 + x_2 + 4x_3$ subject to $\begin{cases} 4x_1 + x_2 - x_3 \le 3 \\ x_1 + x_3 \le 2 \\ 3x_1 + 2x_2 + x_3 \le 15 \end{cases}$ $(x_1, x_2, x_3 \ge 0)$

17. (7 marks) Minimize $C = 2x_1 + 9x_2$ subject to $\begin{cases} 2x_1 + 5x_2 \ge 2 \\ x_1 + 3x_2 \ge 5 \\ 3x_1 + x_2 \ge 1 \end{cases}$ $(x_1, x_2 \ge 0)$

Answers

1. a) $x_1 = -13 - 2t - 8s,\ x_2 = t,\ x_3 = -8 - 2s,\ x_4 = s.$ b) $x_1 = 9,\ x_2 = 5,\ x_3 = 0,\ x_4 = -4.$

2. a) $A^{-1} = \begin{bmatrix} -\frac{8}{19} & -\frac{13}{19} & \frac{11}{19} \\ \frac{1}{19} & \frac{4}{19} & \frac{1}{19} \\ -\frac{12}{19} & -\frac{10}{19} & \frac{7}{19} \end{bmatrix}$, $x = 1,\ y = 2,\ z = -1.$; b) $y = 2$

3. $X = \left(A^T B\right)^{-1} C = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{11}{5} & -\frac{23}{5} \end{bmatrix}$

4. $E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$. *Other possible answers.*

5. I

6. 1) $a \ne 1,\ a \ne 6$; 2) $a = 1$; 3) $a = 6$

7. skew-symmetric

8. -57

9. a) -24; b) $-\frac{729}{2}$; c) 32

10. a) $\pm 2\sqrt{6}$; b) $\left(\frac{12}{7}, -\frac{4}{7}, \frac{8}{7}\right)$; c) 1.9.

11. a) $\frac{\sqrt{14}}{2}$; b) $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$

12. a) $x = 4 + 3t,\ y = 5 + 5t,\ z = t$; b) $x = 2 + 3s,\ y = -1 + 5s,\ z = 3 + s$.

13. a) $\sqrt{5}$; b) $(0,-1,\ 4)$

14. True

15. b) $\frac{1}{\sqrt{2}}$; c) $y+z-4=0$

16. $P=13$, $x_1=0,\ x_2=5,\ x_3=2.$

17. $C=10$, $x_1=5,\ x_2=0.$
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